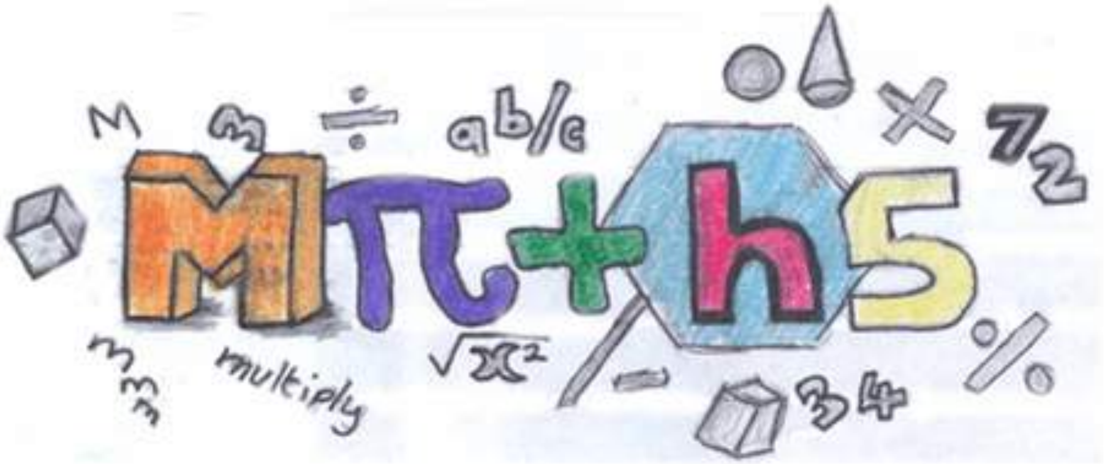
GCSE Mathematics



A/A*

Independent Learning Resource
Learner Work Booklet

Name:

Overview of Resource

Who is this resource for?

- Higher Tier
- Confident on D, C, B topics
- Access to selected A/A* topics

8 key topics organised into 4 learning sections:

(1) **Number:** Surds, Bounds

(2) **Data:** Histograms

(3) **Algebra:** Quadratic Equations, Algebraic Fractions

(4) **Shape:** Vectors, Sine & Cosine Rules, Area of a Triangle

Assessments after each section

Functional Style Questions

Section 1: Number

Surds:

- ❑ Multiplying and dividing surds
- ❑ Simplifying surds
- ❑ Adding and subtracting surds
- ❑ Surd problems

Bounds:

- ❑ Defining upper and lower bound
- ❑ Adding, subtracting, multiplying, dividing measures

Surds

The square roots of **most** numbers cannot be found exactly.

For example, the value of √3 cannot be written exactly as a fraction or a decimal.

The value of √3 is an **irrational number.**

It is often simpler and more mathematically accurate to leave the square root sign in.

These numbers are called **surds.**

Which one of the following is not a surd: √2, √6 , √9 or √14? Why?

Multiplying Surds

Calculate the value of $\sqrt{16} \times \sqrt{16}$

We can think of this as squaring the square root

$\sqrt{3} \times \sqrt{3}$

$$\sqrt{a} \times \sqrt{a} = a$$

$\sqrt{9} \times \sqrt{4}$

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

$\sqrt{2} \times \sqrt{8}$

We can use these two rules backwards when simplifying calculations involving surds

Dividing Surds

Using your calculator, find the value of √20 ÷ √5.

What do you notice about the answer?

$$\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$

What do you think the value of √18 ÷ √2 will be? Why?

Can you find two other surds that follow this pattern?

Match surd laws with results

$\sqrt{a} \times \sqrt{a} =$

$\sqrt{a} \times \sqrt{a} \times \sqrt{a} =$

$\sqrt{a} \div \sqrt{b} =$

$\sqrt{a} \times \sqrt{b} =$

a

$\sqrt{ab}$

$\sqrt{\frac{a}{b}}$

$a\sqrt{a}$

Surd Laws Practice

$\sqrt{6} \times \sqrt{6} =$

$\sqrt{15} =$

$\sqrt{20} =$

$\sqrt{\frac{2}{3}}$

$\sqrt{\frac{4}{5}}$

Surd Laws Practice

$\sqrt{18} \times \sqrt{2} =$

$\sqrt{32} =$

$$\frac{\sqrt{8}}{\sqrt{2}}$$

Surds and Rational Numbers

$\square \times \sqrt{2} = 6$

$\sqrt{27} \times \square = 9$

$\square \div \sqrt{8} = 3$

$\sqrt{48} \div \square = 2$

$\square \div \sqrt{5} = 5$

$\sqrt{32} \times \square = 8$

Simplifying Surds

When solving mathematical problems, we are often required to simplify surds by writing them in the form: $a\sqrt{b}$.

Can you simplify √50 by writing it in the form $a\sqrt{b}$?

Start by finding the largest square number that divides into 50.

We can use this to write:

Adding and Subtracting Surds

Surds can be added or subtracted if the number under the square root sign is the same.

$3\sqrt{7} + 2\sqrt{7}$

Can you simplify the calculation: $\sqrt{27} + \sqrt{75}$?

Step 1: Simplify $\sqrt{27}$ and $\sqrt{75}$

Further Questions

Work out, give your answer in its simplest form

$$\frac{(6-\sqrt{3})(6+\sqrt{3})}{\sqrt{33}}$$

Step 1 –

Step 2 –

Further Questions

Work out, give your answer in its simplest form

$$\frac{(3-\sqrt{2})(2+3\sqrt{2})}{\sqrt{18}}$$

Step 1 –

Step 2 –

Rationalising Denominator

When a fraction has a surd as a denominator, we usually rewrite it so that the denominator is a rational number.

= **rationalising the denominator**

Simplify the fraction $\frac{5}{\sqrt{2}}$

X numerator **and** the denominator by the same number, the value of the fraction remains unchanged.

Rationalising Denominator

Rationalise the denominator $\frac{5}{6 + \sqrt{11}}$

Must multiplying both sides by $6 - \sqrt{11}$

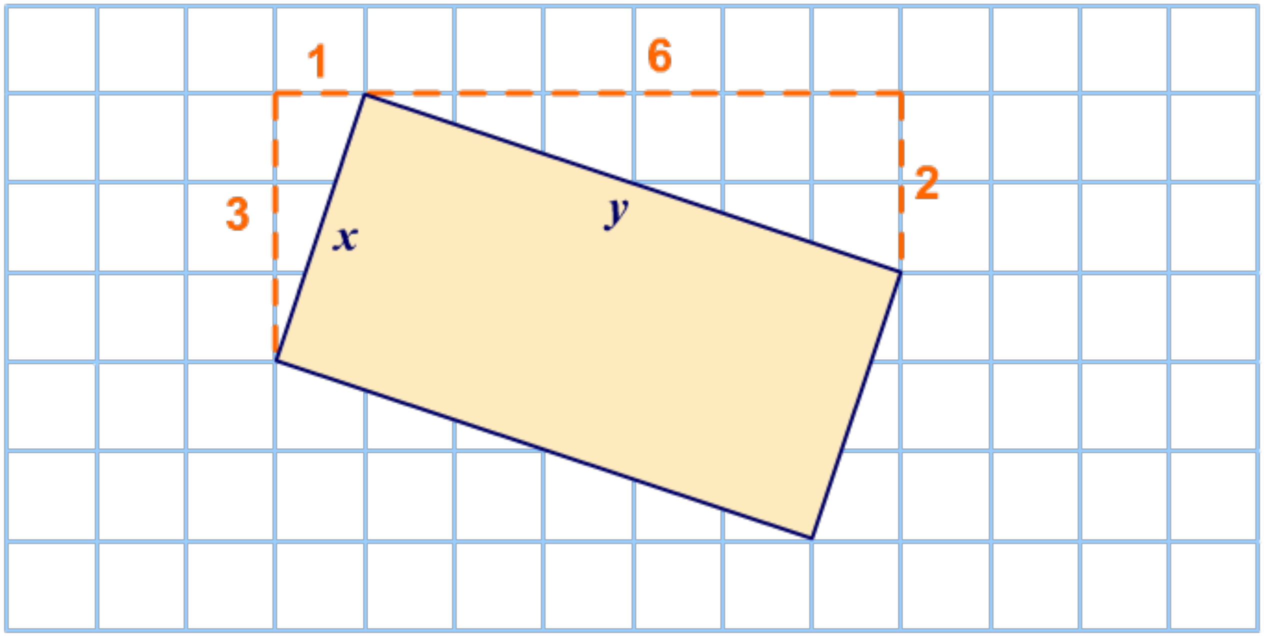
Rationalising Denominator

Rationalise the denominator $\frac{8}{4 - \sqrt{10}}$

Must multiplying both sides by $4 + \sqrt{10}$

Problem Solving

The following rectangle has been drawn on a centimetre grid.



Use the given lengths to find the length (y) and width (x) of the rectangle.

Find its perimeter and area in surd form.

Exam Question

A large rectangular piece of card is $(\sqrt{5} + \sqrt{20})$ cm long and $\sqrt{8}$ cm wide.

A small rectangle $\sqrt{2}$ cm long and $\sqrt{5}$ cm wide is cut out of the piece of card.

Express the area of the card that is left as a percentage of the area of the large rectangle

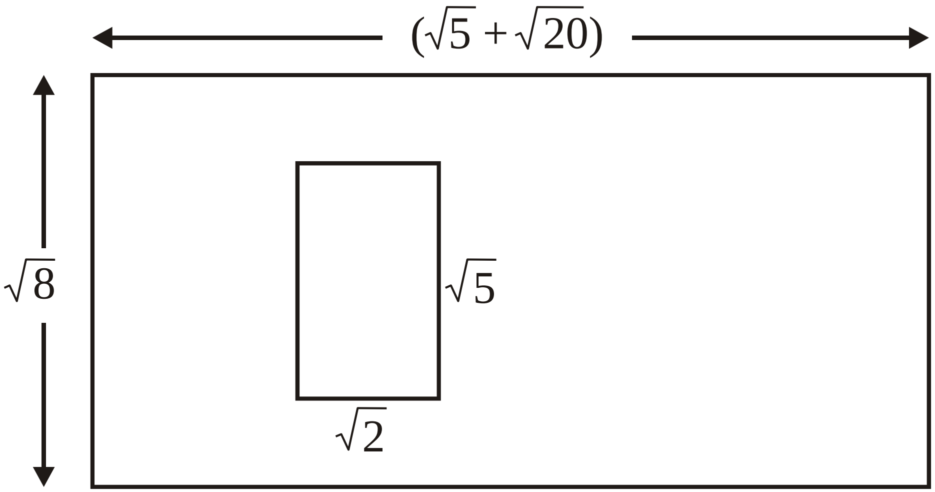


Diagram **NOT** accurately drawn

4 marks

Upper & Lower Bounds

Applying the conventions of rounding

My friend´s height is 146cm to the nearest cm

The *most* this could be before being rounded *down* is:

This value is called the lower bound…

The *least* this could be before being rounded *up* is:

… and this value is called the upper bound.

Upper & Lower Bounds

Find the upper and lower bounds of each of these numbers:

80cm to the nearest cm

1.4 to 1 decimal place

120 to the nearest integer

32.43 to 2 decimal places

32.40 to 2 decimal places

40 to the nearest ten

70m to the nearest cm

Operations - Bounds

ADDING

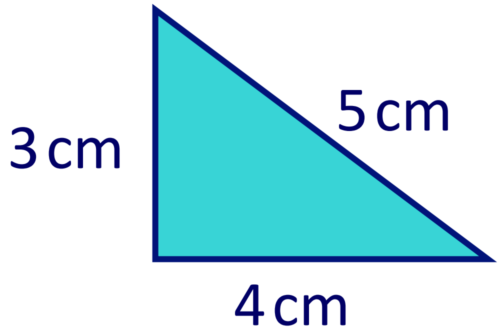
SUBTRACTING

MULTIPLYING

DIVIDING

Adding Measures

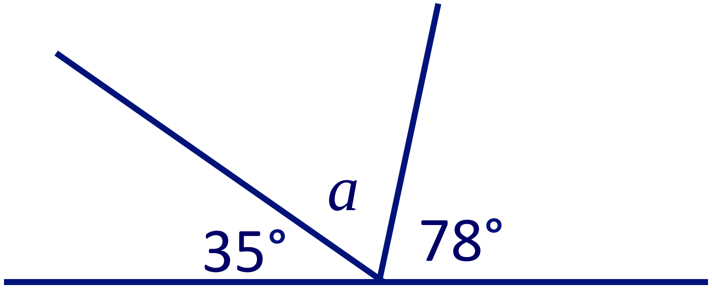
The following triangle has sides of length 3 cm, 4 cm and 5 cm.



What is the upper and lower bound for the perimeter?

Adding Measures

The angles in the following diagram are rounded to the nearest degree:

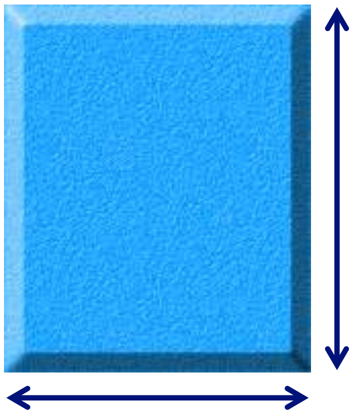


What is the upper bound and lower bound for angle a?

Multiplying Measures

The dimensions of a small tile are given as 15 mm by 18 mm.

Calculate the upper bound and lower bound for the area of the tile



Dividing Measures

A boy runs 200 metres in 27.8 seconds.

The distance is given to the nearest metre and the time is given to the nearest tenth of a second.

What is his greatest possible average speed to 2 decimal places?

Exam Question

Work out the upper and lower bound for the value of d.
Give you answer correct to 2 decimal places

$$d = \frac{2.6K}{F^2 \sin x}$$

F = 1.4 correct to 1 decimal place
K = 3.20 correct to 2 decimal places
x = 30 correct to the nearest integer

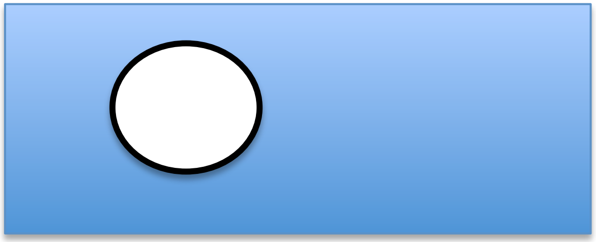
DIVIDING

$L = \frac{L}{U}$

$U = \frac{U}{L}$

Exam Question

A circular disc is cut out of a rectangular piece of metal. Calculate the maximum and minimum area of metal left



The length of the rectangle is 12cm and the width is 8cm.
The radius of the circle is 2cm.
All measures are correct to the nearest cm.
Take π to be 3.14

Assessment 1: Number

10 minutes

Section 2: Data

Histograms:

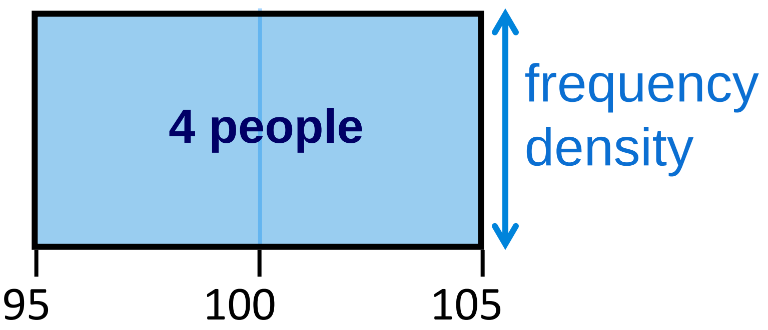
- ❑ Calculating frequency density
- ❑ Constructing a histogram
- ❑ Interpreting a histogram

Histograms

Histograms are not bar charts. Why?

1) The class intervals may be unequal
2) The y axis is frequency DENSITY
3) Areas of the bars (not the height) represent frequency
4) A* topic

$$\text{Frequency Density (height of the bar)} = \frac{\text{frequency}}{\text{width of interval}}$$



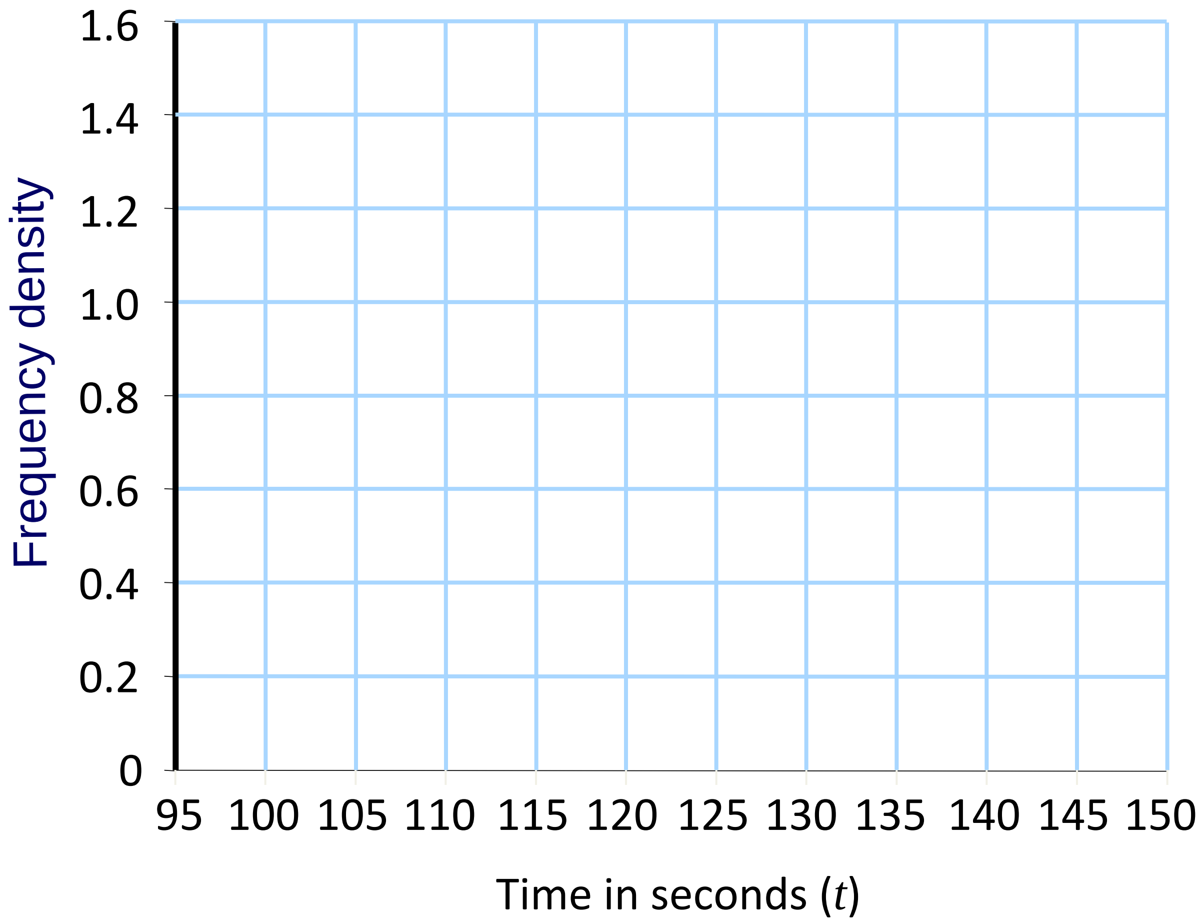
Calculating Frequency Density

Time in seconds	Frequency	Frequency density
$95 \leq t < 100$	8	-------
$100 \leq t < 105$	5	-------
$105 \leq t < 115$	8	-------
$115 \leq t < 130$	12	-------
$130 \leq t < 150$	2	-------

$$\text{Frequency density} = \frac{\text{frequency}}{\text{width of interval}}$$

Enter the answers (to 1 d.p) on the dotted lines.

Drawing the Histogram



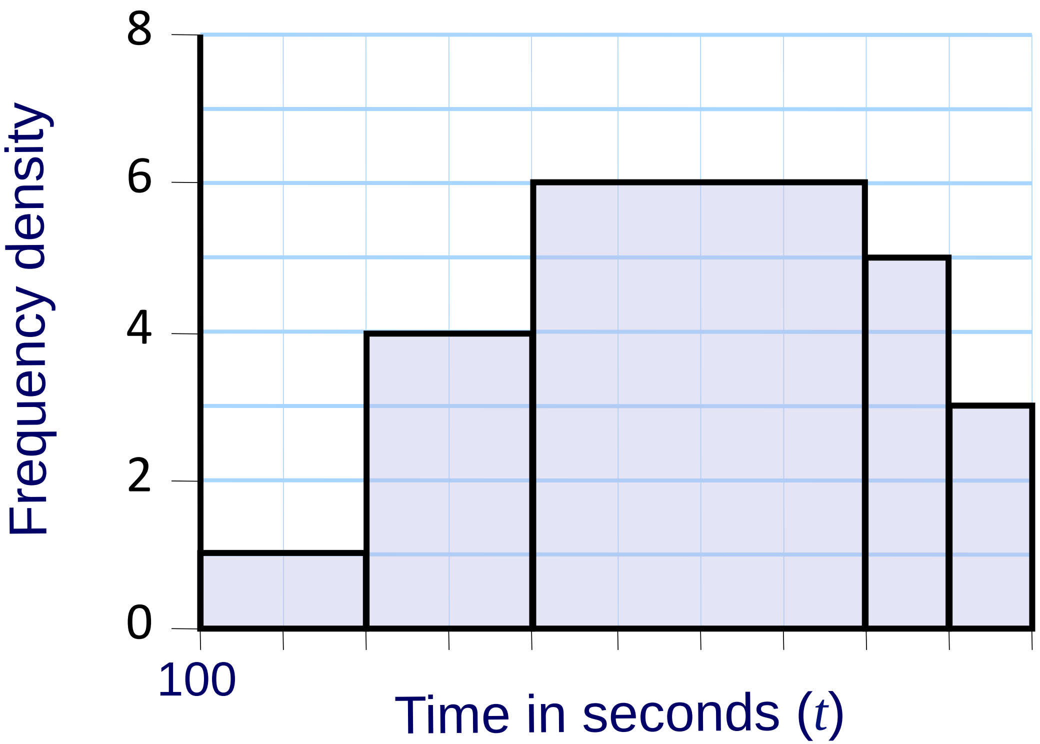
Complete the table

We can rearrange the formula:

frequency = frequency density × width of interval

Time in seconds	Frequency density	Frequency
$95 \leq t < 105$		**4**
$105 \leq t < 110$	**1.2**	
$110 \leq t < 115$		**7**
$115 \leq t < 120$	**2.2**	**11**
$120 \leq t < 125$	**1.2**	

Calculating Class Intervals



The first bar in the histogram represents 40 people.

The lowest time recorded in the race was 100 seconds.

Work out the scale along the bottom and the frequencies for each interval.

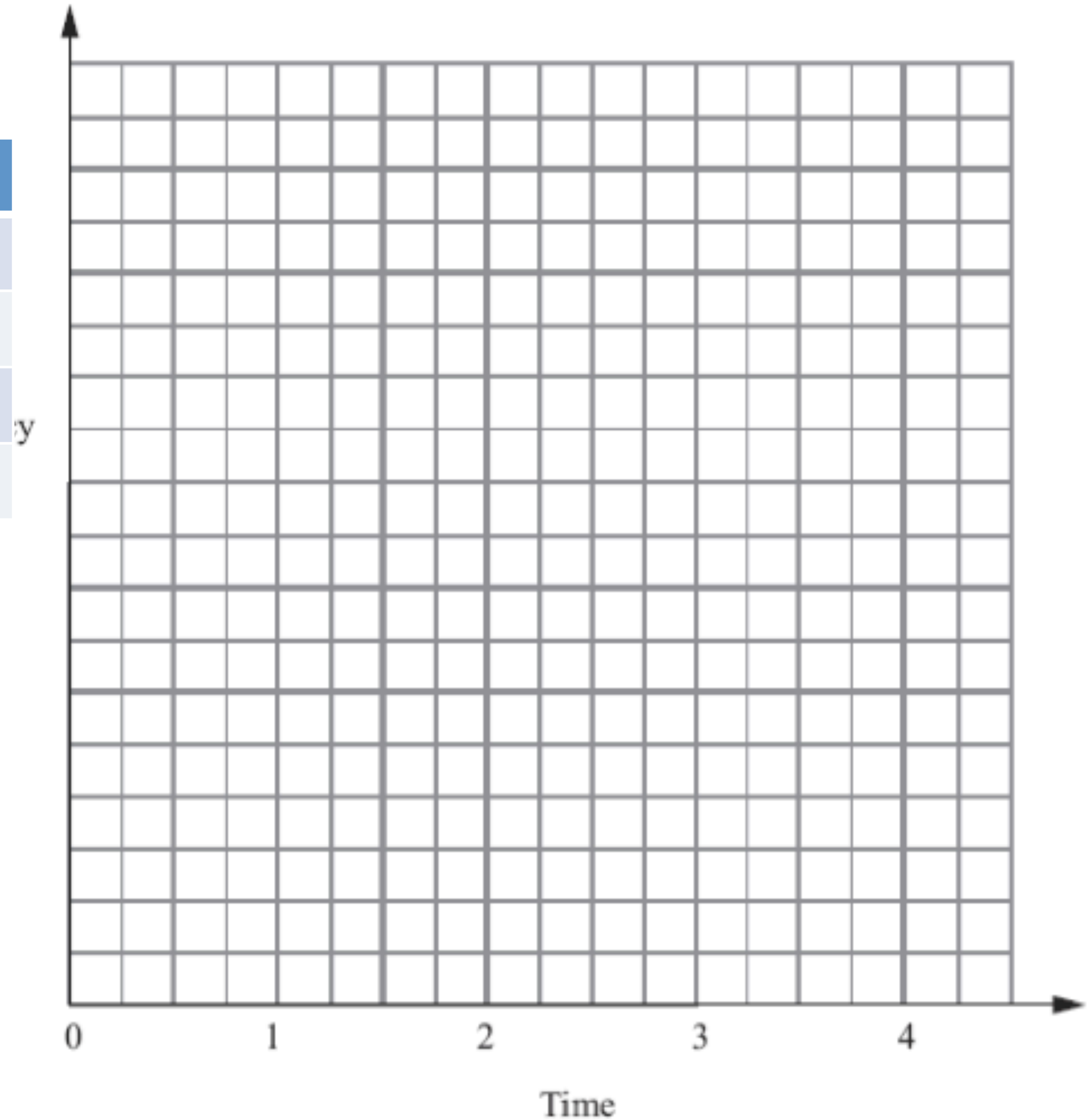
Frequency	Frequency density	Width of interval	Class interval
40	1	------	---- $\leq t <$ ----
------	4	40	---- $\leq t <$ ----
480	------	80	---- $\leq t <$ ----
100	------	20	---- $\leq t <$ ----
60	3	------	---- $\leq t <$ ----

Exam Question

The table shows information about the total times that 35 students spent using their mobile phones one

Time (h hours)	Frequency
0 ≤ h < ½	8
½ ≤ h < 1	7
1 ≤ h < 2	11
2 ≤ h < 4	9

On the grid draw a histogram to represent this data



Assessment 2: Data

10 minutes

Section 3: Algebra

Solving Quadratic Equations:

- ❑ Factorisation
- ❑ Completing the square
- ❑ The quadratic formula

Algebraic Fractions:

- ❑ Simplifying algebraic fractions
- ❑ Adding and subtracting algebraic fractions
- ❑ Multiplying and dividing algebraic fractions
- ❑ Solving equations involving algebraic fractions

Quadratic Equations

If the highest power is x^2 the equation is **quadratic.**

Basic form of a quadratic: $ax^2 + bx + c = 0$

Example: $x^2 + 4x - 45 = 0$

One method to solve is **factorization.**

$(x + ...)(x + ...) = 0$

NEED: two integers that

Factorisation

$$x^2 + 6x + 8 = 0$$

Factorisation

$x^2 - 2x - 15 = 0$

$x^2 + 7x + 12 = 0$

$x^2 + 5x - 6 = 0$

Factorisation – coefficient x not 1

$3x^2 - x - 2 = 0$

Factorisation – coefficient x not 1

$2x^2 + 5x + 2 = 0$

Completing the Square

Another method to solve a quadratic is by "completing the square"

Step 1: Divide the coefficient of *x* by 2 ------number that goes inside the bracket with x

Step 2: Compare constant with constant in equation, adjust as necessary

$$x^2 + 8x + 12 = 0$$

Completing the Square

Complete the square for $x^2 + 12x - 5$.

What would be the next step to SOLVE $x^2 + 12x - 5 = 0$?

Completing the Square

When the coefficient of x^2 is not 1, quadratic equations in the form $ax^2 + bx + c$ can be rewritten in the form $a(x + p)^2 + q$.

Complete the square for $2x^2 + 8x + 3$.

Start by factorizing the first two terms by dividing by 2:

Completing the Square

Complete the square for $5 + 6x - 3x^2$.

Start by factorizing the the terms containing x's by –3.

Solving by completing the square

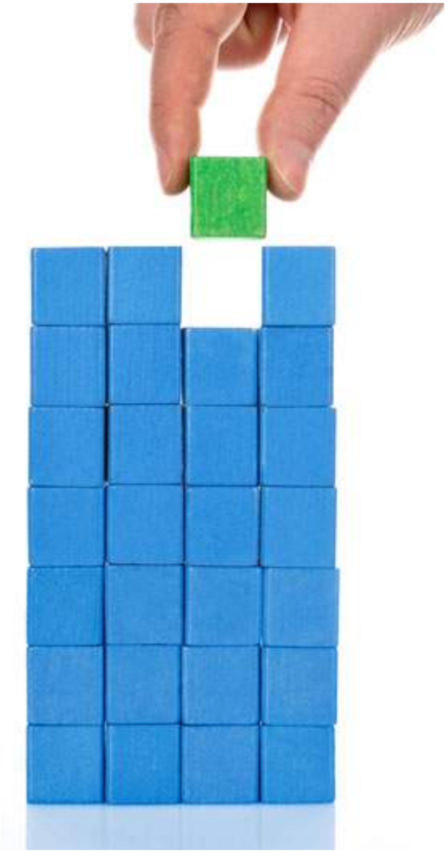
Solve the equation $x^2 + 8x + 5 = 0$ by completing the square.
Write the answer to 3 decimal places.

$$x^2 + 8x + 5 = 0$$

Solving by completing the square

Solve the equation $2x^2 - 4x + 1 = 0$ by completing the square. Write the answer to 3 decimal places.

$$2x^2 - 4x + 1 = 0$$



Quadratic Formula

Any quadratic equation of the form,

$$ax^2 + bx + c = 0$$

can be solved by substituting the values of a, b and c into the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

On formula sheet

Quadratic Formula

$3x^2 + 4x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Formula

$2x^2 + 7x - 4 = 0$

Give your answer correct to 3 significant figures

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Algebraic Fractions

NORMAL fractions, but contain algebraic expressions in the numerator/denominator.

$\frac{3x}{4x^2}$ and $\frac{2a}{3a + 2}$ are two examples of algebraic fractions.

The rules that apply to numerical fractions also apply to algebraic fractions.

For example, if we multiply or divide the numerator and the denominator of a fraction by the same number or term, we produce an equivalent fraction.

$$\frac{3x}{4x^2} =$$

Simplifying Algebraic Fractions

$$\frac{6ab}{3ab^2}$$

$$\frac{5x + 2x^2}{3x}$$

$$\frac{3ab + 12b^2}{ab}$$

Simplifying by Factorising

Sometimes need to factorise the numerator/denominator before simplifying:

$$\frac{2a + a^2}{8 + 4a}$$

$$\frac{6x + x^2}{x^2 + 8x + 12}$$

+/- Algebraic Fractions

Same method that we use for numerical fractions.

$$\frac{1}{a} + \frac{2}{b}$$

Need a common denominator before +/-

General rule +/- algebraic fractions is:

$$\frac{a}{b} +/- \frac{c}{d} = \frac{ad +/- bc}{bd}$$

$$\frac{p}{3} - \frac{q}{2}$$

+/- Algebraic Fractions

$$\frac{3}{x} + \frac{y}{2}$$

$$\frac{2 + p}{4} - \frac{3}{2q}$$

Manipulating Algebraic Fractions

If two terms are added or subtracted in the numerator, the fraction can be split in two over a common denominator.

For example, $$\frac{1+2}{3} = \frac{1}{3} + \frac{2}{3}$$

$\frac{a+b}{c}$ can be written as $\frac{a}{c} + \frac{b}{c}$

If two terms are added or subtracted in the denominator of a fraction, we cannot split the fraction into two.

For example, $$\frac{3}{1+2} \neq \frac{3}{1} + \frac{3}{2}$$

$\frac{c}{a+b}$ *cannot* be written as $\frac{c}{a} + \frac{c}{b}$

Manipulating Algebraic Fractions

Algebraic fractions can be multiplied using the same rules that are used for numerical fractions.

$$\frac{3p}{4} \times \frac{2}{(1-p)}$$

In general, the rule for multiplying algebraic fractions is:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Dividing Algebraic Fractions

Algebraic fractions can be divided using the same rules that are used for numerical fractions.

$$\frac{2}{3y-6} \div \frac{4}{y-2}$$

In general, the rule for dividing algebraic fractions is:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Multiplying & Dividing Fractions

$\frac{a + b}{c} \times \frac{cd}{a} =$

$\frac{12c^2}{(a + b)^2} \div \frac{4}{a + b} =$

$\frac{8}{(a + b)^3} \times \frac{a + b}{4} =$

$\frac{a + b}{5} \div \frac{2}{(a + b)^2} =$

$\frac{(a + b)^3}{10}$

$\frac{d(a + b)}{a}$

$\frac{3c^2}{a + b}$

$\frac{2}{(a + b)^2}$

Solving Equations With Fractions

$$\frac{x}{2} + \frac{x}{3} = \frac{25}{2}$$

Solving Equations With Fractions

$$\frac{3}{x} - \frac{3}{2x} = \frac{3}{8}$$

Solving Equations With Fractions

$$\frac{5}{x+3} + \frac{3}{2x-1} = 2$$

Assessment 3: Algebra

10 minutes

Section 4: Shape

Vectors:

- ❑ Vector notation
- ❑ Vector calculations
- ❑ Using vectors to solve problems

Triangles:

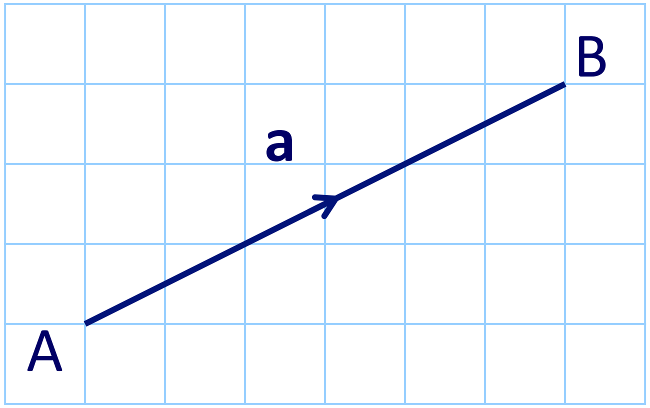
- ❑ Sine rule
- ❑ Cosine rule
- ❑ Area of a triangle

Vectors

A vector is a quantity that has both size (or magnitude) & direction.

A vector can be represented using a line segment with an arrow on it.

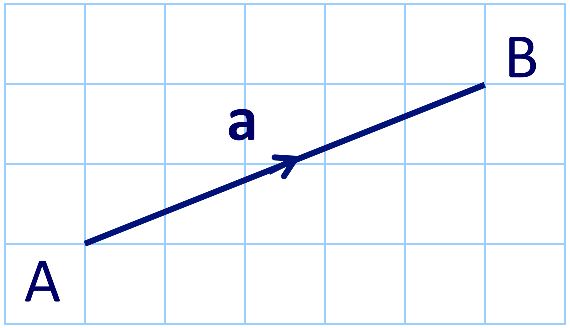
For example,



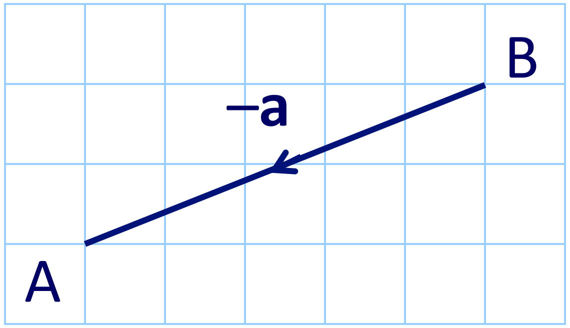
We can write this vector as:

The Negative of a Vector

Here is the vector $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$



Suppose the arrow went in the opposite direction:



How can we describe this vector?

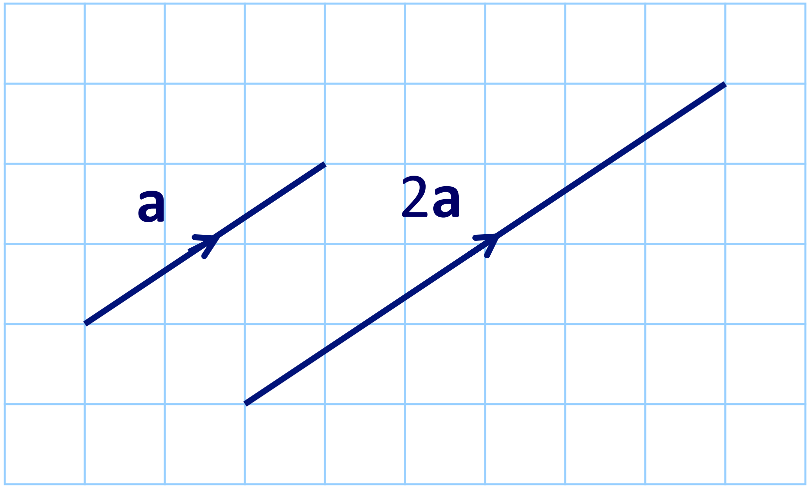
We can describe this vector as:

Multiplying Vectors by Scalars

A scalar quantity has size but not direction.

A scalar quantity can be represented by a single number.

A vector can be multiplied by a scalar.



The vector 2**a** has the same direction but is twice as long.

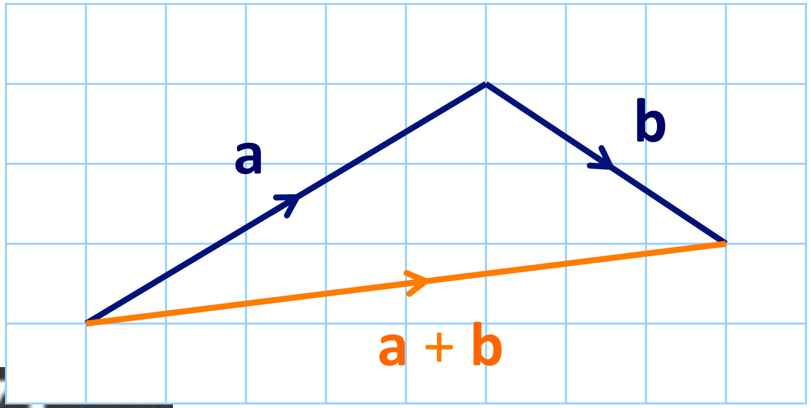
Adding Vectors

Adding two vectors is equivalent to applying one vector followed by the other. For example,

Suppose $\mathbf{a} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Find a + b

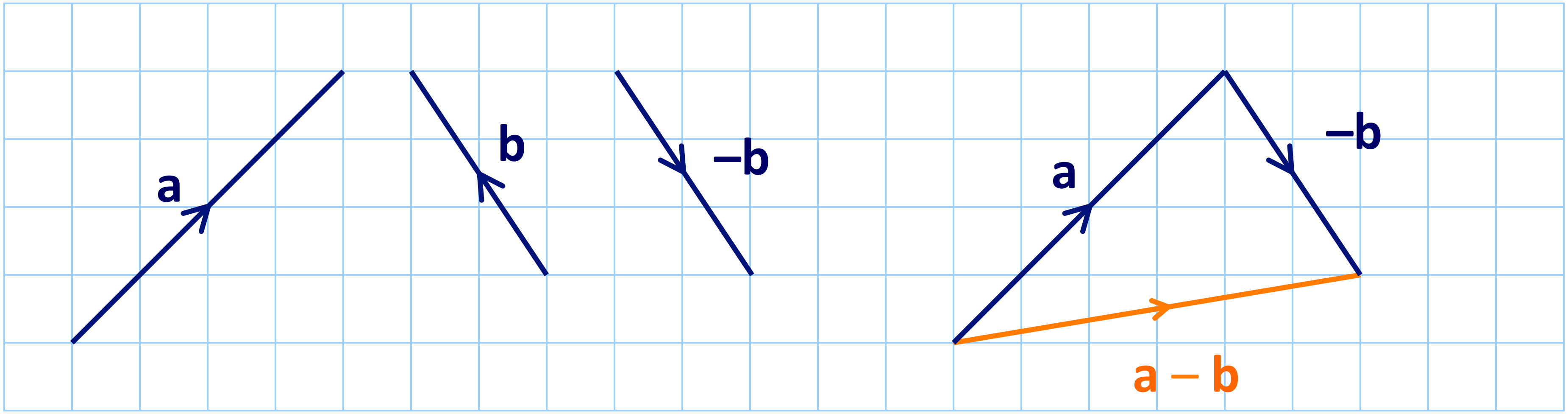
We can represent this addition in the following diagram:



Subtracting Vectors

To show this subtraction in a diagram, we can think of **a** – **b** as **a** + (–**b**).

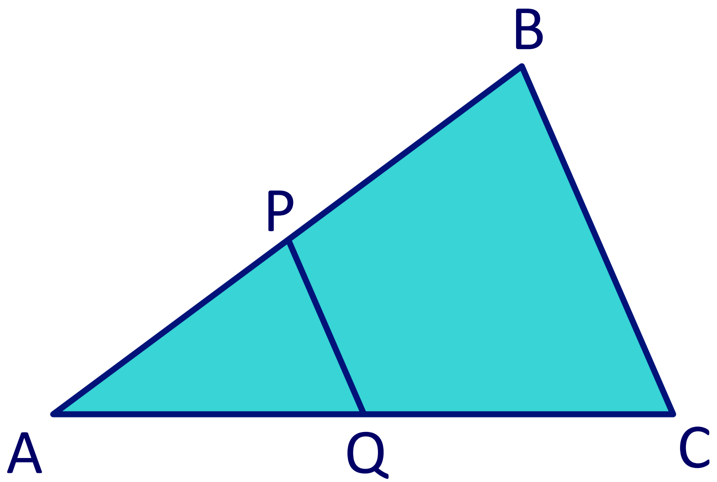
$$\mathbf{a} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$



Using vectors to solve problems

Vectors can be used to solve many problems involving physical quantities such as force and velocity and can also be used to prove geometric results.

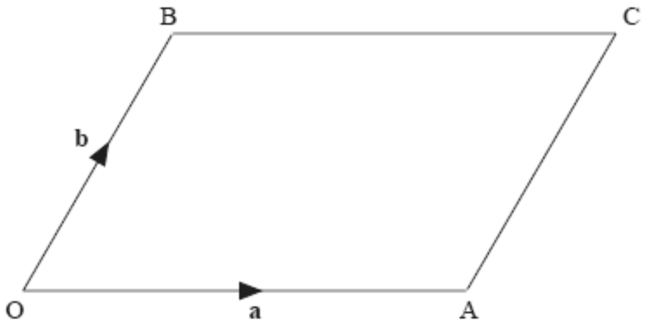
For example, suppose we have a triangle ABC as follows:



The line PQ is such that P is the mid-point of AB and Q is the mid-point of AC.

Use vectors to show that PQ is parallel to BC and that the length of BC is double the length of PQ.

Exam Question



Write the following vectors in terms of **a** and **b:**

BC

CA

AB

OC

X is the midpoint of OA and Y is the midpoint of OB. Find in terms of **a** and **b:**

OX

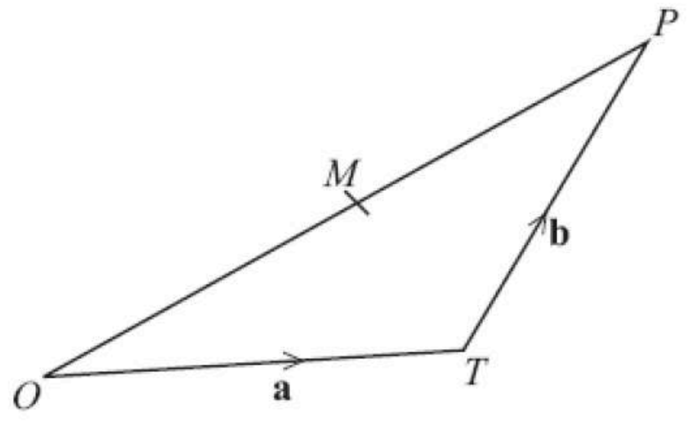
OY

XY

Exam Question

OTP is a triangle.
M is the midpoint of OP.

Write the following vectors in terms of **a** and **b:**



OM

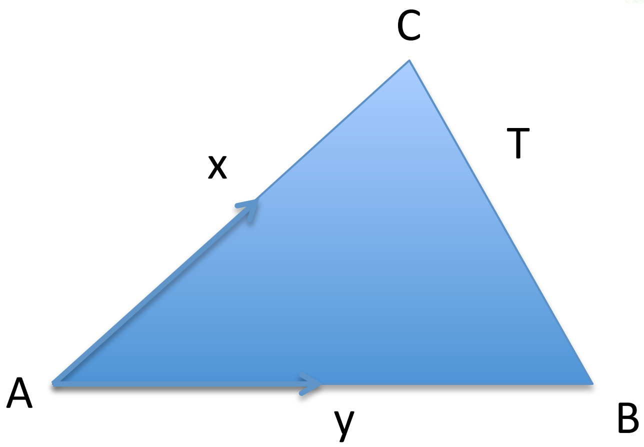
TM

Exam Question

ABC is a triangle

AC = x
AB = y

CT: TB = 1:2

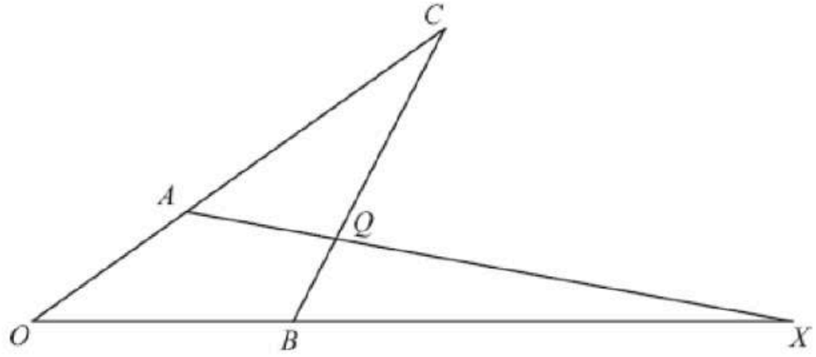


Write the following vectors in terms of **x** and **y:**

BC

AT

Exam Question



In the diagram,

OA 0 4a and OB = 4b

OAC, OBX, and OQC are all straight lines

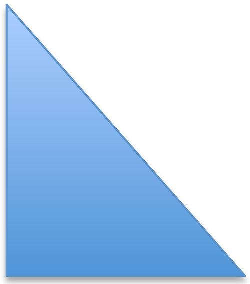
AC = 2OA and BQ: QC = 1:3

Write the following vectors in terms of **a** and **b:**

BC

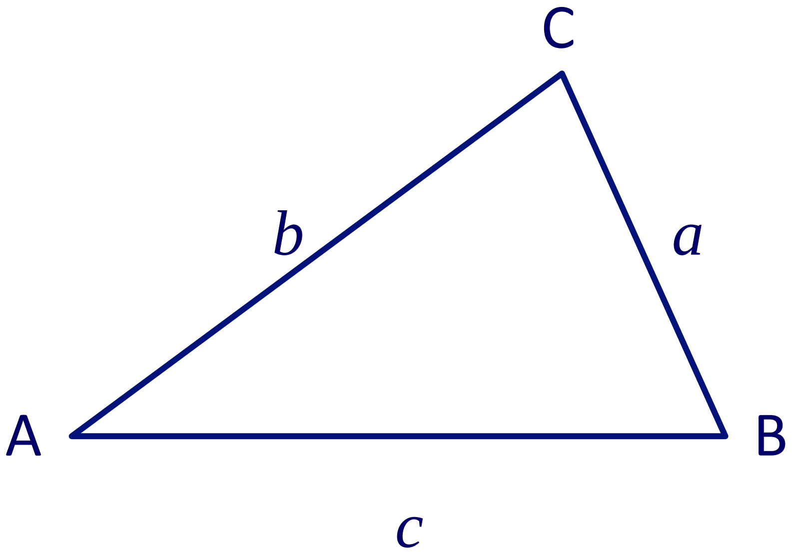
AQ

The Sine and Cosine Rules

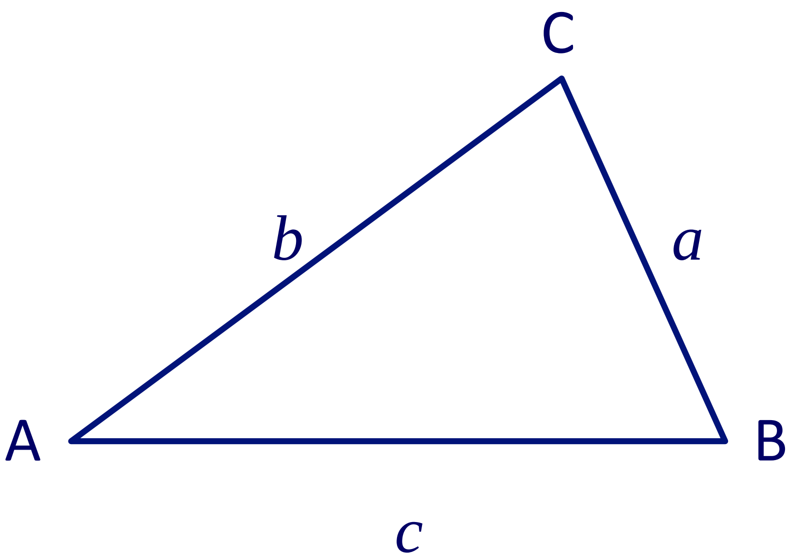


Right angle triangles use **SOHCAHTOA**

NON right angle triangles use **Sine and Cosine Rules**



The Sine and Cosine Rules



WHICH rule?

Sine

2 angles + side

2 sides + non included angle

Cosine

2 sides + included angle

3 sides + NON angle

Sine Rule: Finding Sides

$$\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}$$

Finding Angles

$$\frac{\text{Sin A}}{a} = \frac{\text{Sin B}}{b} = \frac{\text{Sin C}}{c}$$

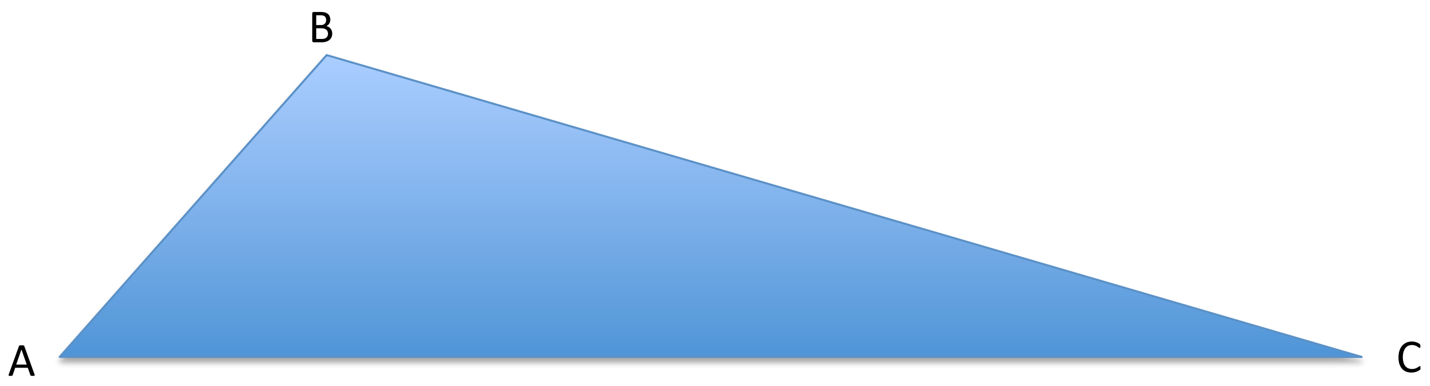
Cosine Rule: Finding Sides

$$a^2 = b^2 + c^2 - 2bc \cos \text{A}$$

Finding Angles

$$\cos \text{A} = \frac{b^2 + c^2 - a^2}{2bc}$$

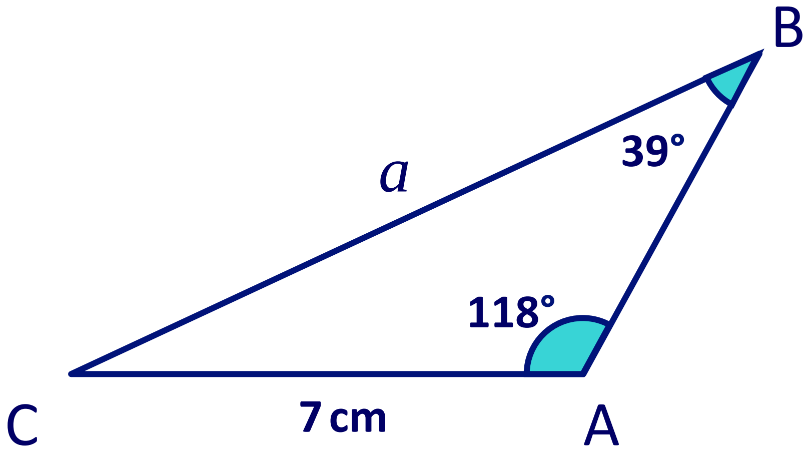
Sine Rule – Finding Lengths



AB = 5cm
AC = 12cm
Angle ACB is 21^o
Angle BAC is 52^o
Find length BC
Give your answer correct to 3 significant figures.

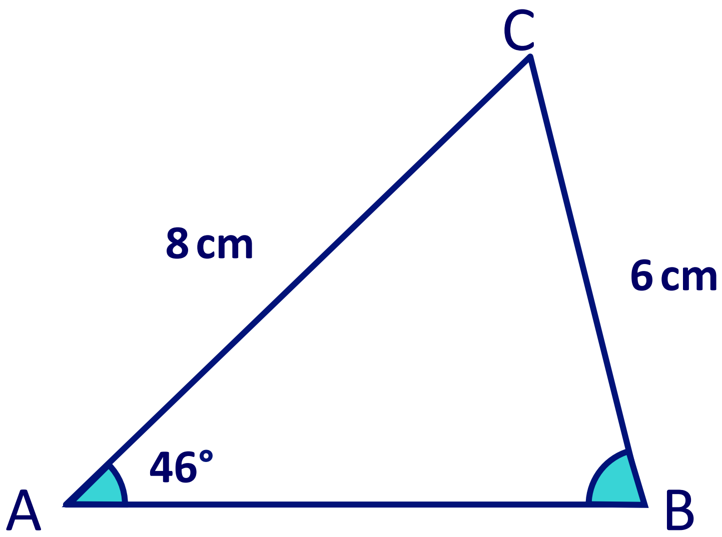
Sine Rule – Finding Lengths

Find the length of side *a*.



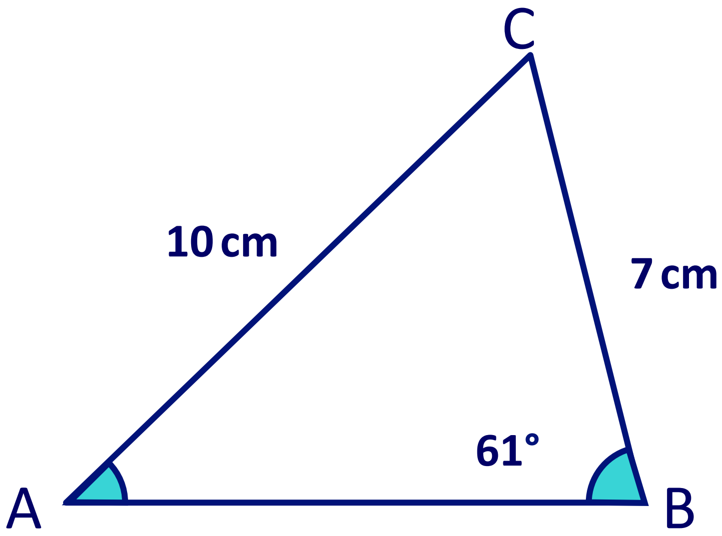
Sine Rule – Finding Angles

Find the angle at B.



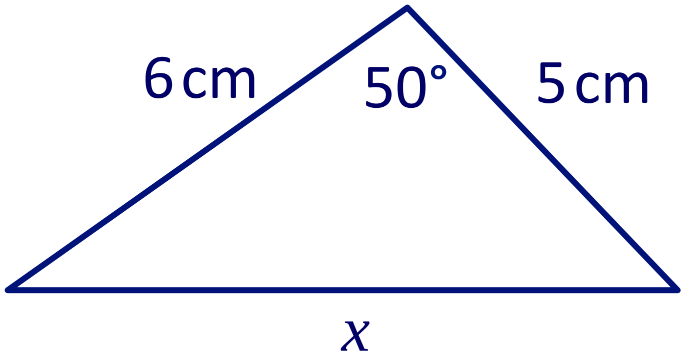
Sine Rule – Finding Angles

Find the angle at B.



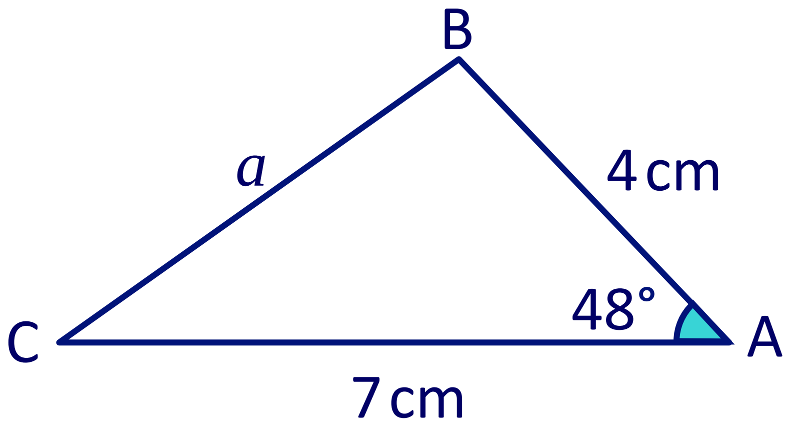
Cosine Rule – Finding Lengths

Find the length of side *x*



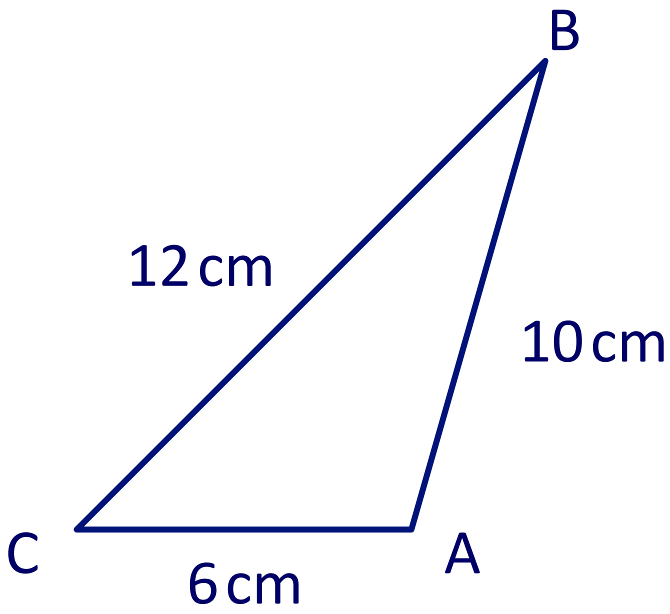
Cosine Rule – Finding Lengths

Find the length of side a.



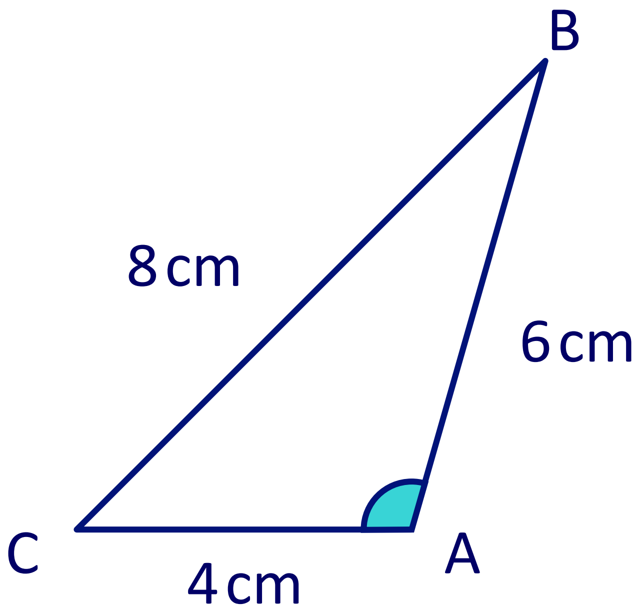
Cosine Rule – Finding Angles

Find the size of the angle at B



Cosine Rule – Finding Angles

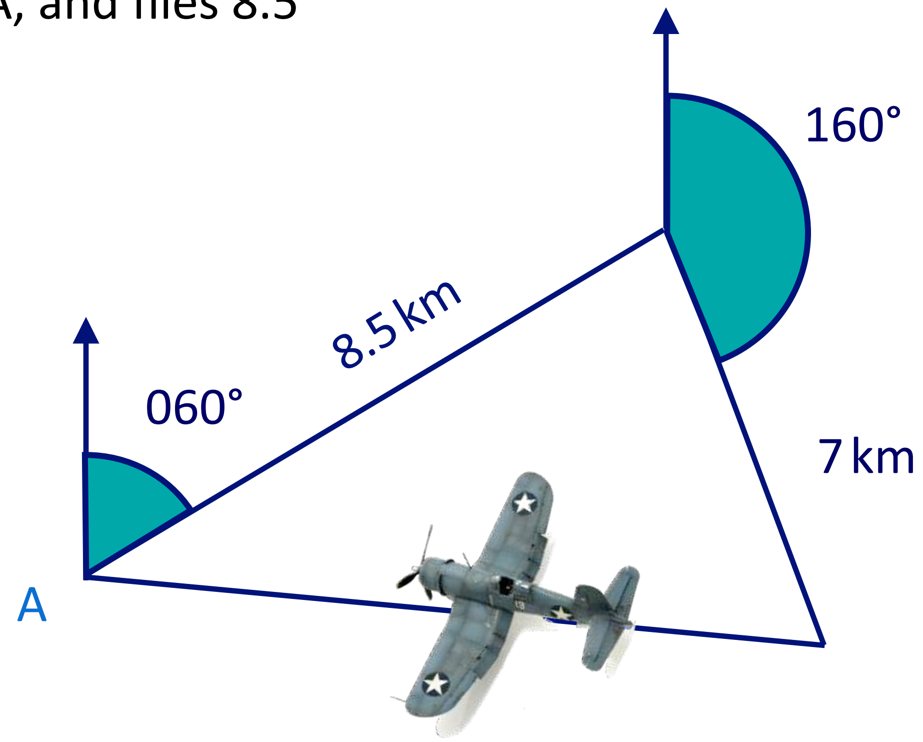
Find the size of the angle at A.



Problem Solving

An aeroplane starts at the airport, A, and flies 8.5 km on a bearing of 060°.

It then gets blown off course on a bearing of 160° for 7 km.



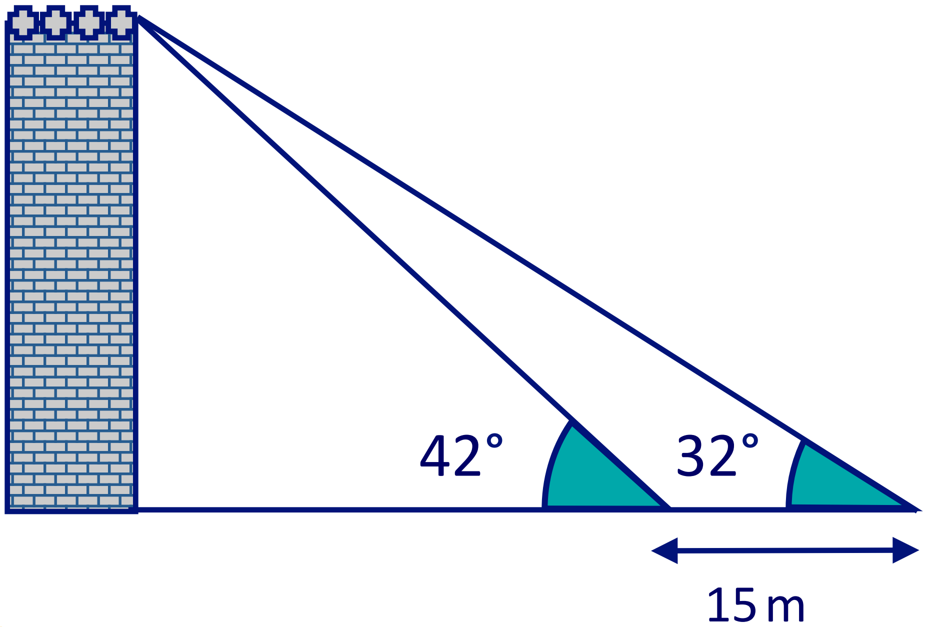
The captain decides to return home.
How far does she have to fly and in what direction?

Problem Solving

A surveyor is trying to calculate the height of a tower.

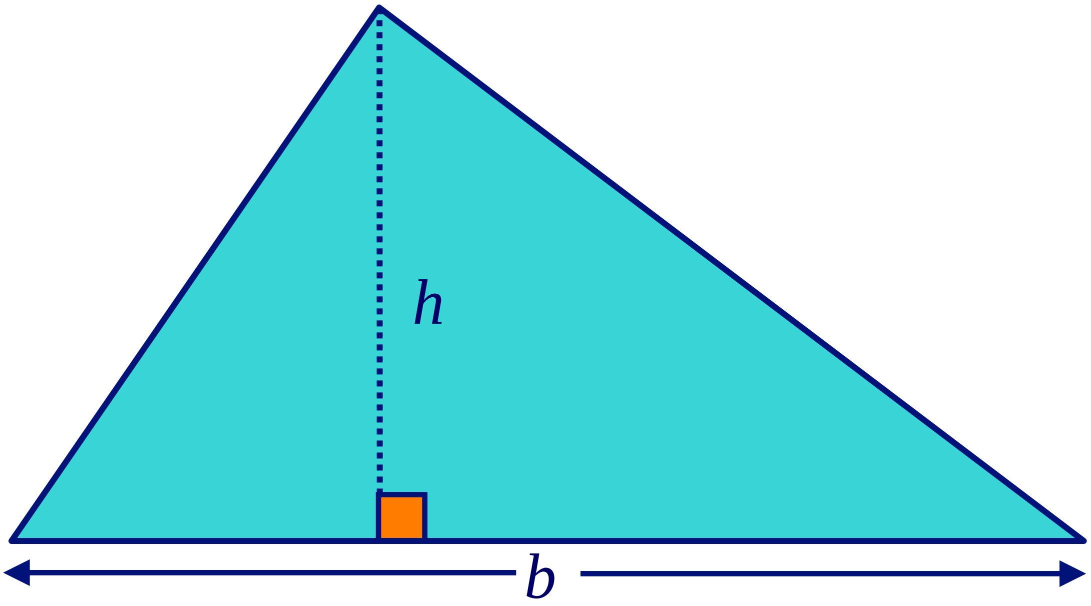
He measures the angle of elevation at 32°.

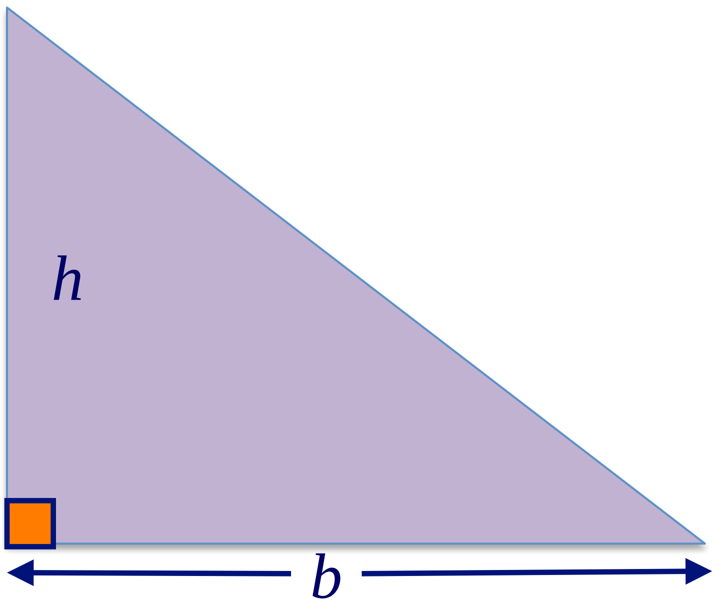
He then moves closer by 15m, and the angle is now 42°.



Find the height of the tower.

Area of Triangle

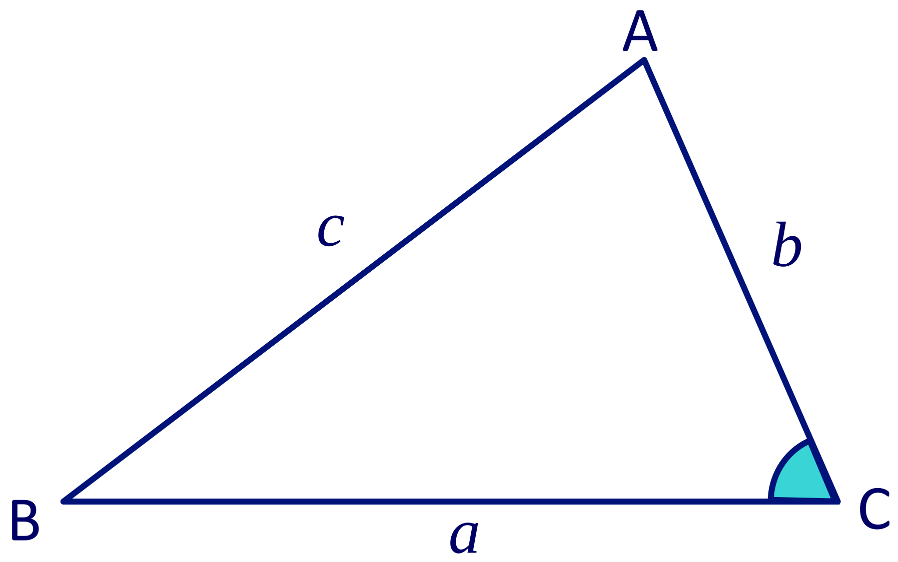




Area of a triangle = $\frac{1}{2}bh$

Area of Triangle ½ a b sinC

The area of a triangle is equal to half the product of two of the sides and the sine of the included angle.

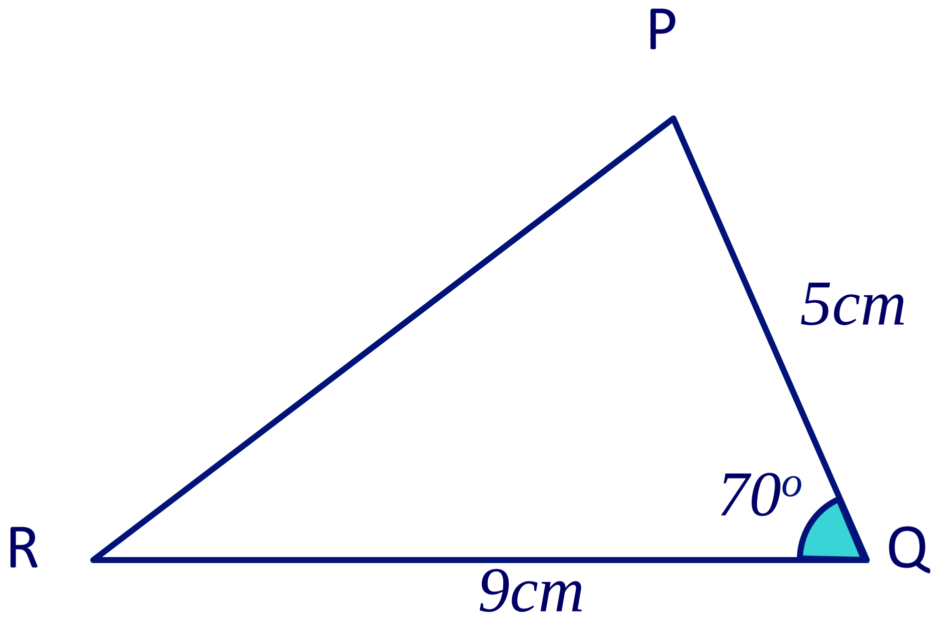


Need 2 sides + included angle

Area of triangle ABC = $\frac{1}{2}ab \sin C$

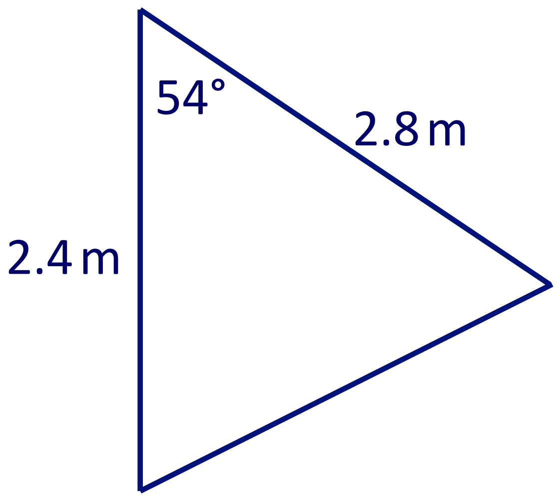
On formula sheet

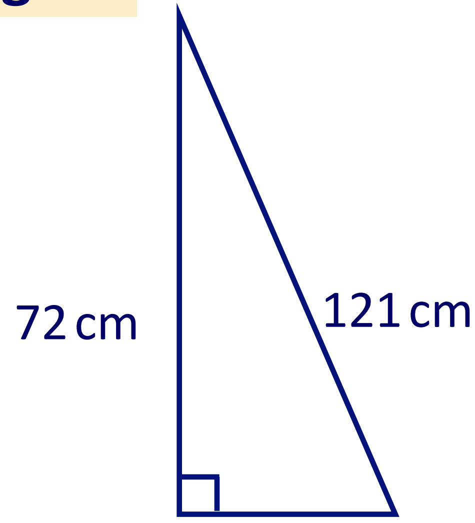
Area of Triangle ½ a b sinC

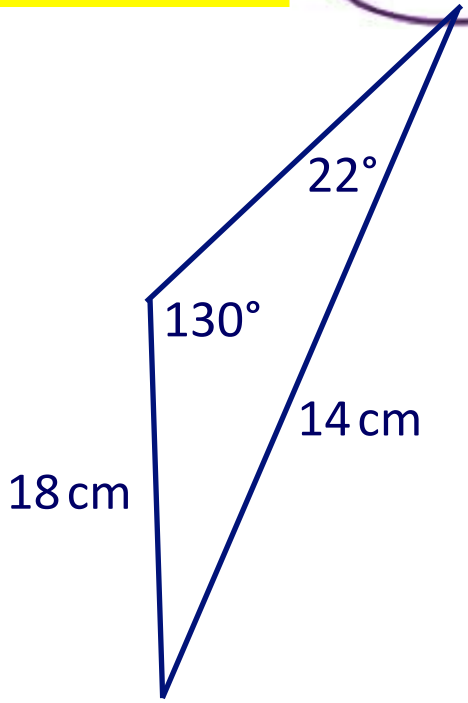


Area of Triangle ½ a b sinC

Find the areas of the 3 triangles.







Assessment 4: Shape

boardworks

the PiXL club

10 minutes